2007 International Symposium on Nonlinear Dynamics (2007 ISND) IOP Publishing
Journal of Physics: Conference Series **96** (2008) 012164 doi:10.1088/1742-6596/96/1/012164

# Nonlinear Focusing Manakov Systems by Variational Iteration Method and Adomian Decomposition Method

**N. H. Sweilam [1] *, M. M. Khader [2] and R. F. Al-Bar [3]**

[1] Department of Mathematics, Faculty of Science, Cairo University, Giza-Egypt.

[2] Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt.

[3] Department of Mathematics, Faculty of Science, Umm Al-Qura University, KSA.

Email [1] : n_sweilam@ yahoo.com

**Abstract.** In this paper, the variational iteration method (VIM) and the Adomian decomposition method (ADM) are applied to solve numerically the focusing Manakov systems of coupled nonlinear Schrödinger equations. The accuracy of the methods are verified by ensuring that the conserved quantities remain almost constant. The results show that VIM is much easier, more convenient, more stable and efficient than ADM.

## 1. Introduction

The focusing Manakov systems in 1+1 dimensions (i.e., one space and one time dimension) are coupled systems of nonlinear Schrödinger equations models the propagation of the average fields of a polarized wave in a randomly birefringent optical fiber. The qualitative natures of the unstable manifolds of linearly unstable plane wave solutions have been examined in [14] and [17]. For more details on the coupled nonlinear Schrödinger equations and the focusing Manakov systems see [6], [11], [21], [24] and the references cited therein. Recently much attentions have been devoted to the numerical methods which do not require discretization of space-time variables or linearization of the nonlinear equations, among which are the Adomian decomposition method (ADM) ( [1], [4], [7], [16] ) and the variational iteration method (VIM) which is suggested by Ji- Huan He ([2], [3], [8]-[10], [19] and the reference cited therein) and it based on Lagrange multiplier method. Many authors are pointed out that VIM has merits over other methods and can overcome the difficulties arising in calculation of Adomian polynomials in the ADM (see [5], [13], [18] and the references therein).

The main aim of this paper is to develop VIM and ADM to solve the following Manakov system of nonlinear coupled one dimension partial differential equations [15], [20], [21]:

$$i\,u_t + 0.5\,u_{xx} + q\,(|u|^2 + |v|^2)\,u = 0\ , \qquad x \in R,\ t \geq 0\ . \tag{1.1}$$

* To whom any correspondence should be addressed

$$i\,v_t + 0.5\,v_{xx} + q\,(\,|u|^2 + |v|^2)\,v = 0, \tag{1.2}$$

with initial conditions $u(x,0) = u^0(x)$, $v(x,0) = v^0(x)$ and homogenous boundary conditions. There has been a lot of previous work on the solitary wave equations to system (1.1)-(1.2) under the infinite boundary conditions $u \to 0$, $v \to 0$ at $|x| \to \infty$ (see [20]-[21] and the references therein). An important goal of the present work is to show that the developed VIM which remarkable merits over other methods is applicable to solve numerically the focusing Manakov systems, i.e., $q = +1$, in the above equations.

## 2. Implementation of VIM and ADM

### 2.1. Implementation of VIM

Consider the system (1.1)-(1.2) and by using VIM in the region $R = [x_L < x < x_R] \times [t > 0]$ with its boundary $\partial R$ which consists of the ordinates $x = x_L, x = x_R$, we can construct the correction variational functional equations as following:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\tau)[\,u_{n\tau} - 0.5\,i\,\hat{u}_{nxx} - i\,(\,|\hat{u}_n|^2 + |\hat{v}_n|^2)\,\hat{u}_n]\,d\tau \quad , \tag{3.1}$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(\tau)[\,v_{n\tau} - 0.5\,i\,\hat{v}_{nxx} - i\,(\,|\hat{u}_n|^2 + |\hat{v}_n|^2)\,\hat{v}_n]\,d\tau \quad , \tag{3.2}$$

where $\lambda_1$ and $\lambda_2$ are the general Lagrange multipliers, $\hat{u}_{nxx}$, $\hat{u}_n$, $\hat{v}_{nxx}$, $\hat{v}_n$ denote restricted variations
i.e., $\delta\,\hat{u}_{nxx} = \delta\,\hat{u}_n = \delta\,\hat{v}_{nxx} = \delta\,\hat{v}_n = 0$. Making the above correction functional stationary, we can obtain the following stationary conditions:

$$\dot{\lambda}_1(\tau) = 0\,,\quad \dot{\lambda}_2(\tau) = 0\;,\quad 1 + \lambda_1(\tau)|_{\tau = t} = 0\;,\quad 1 + \lambda_2(\tau)|_{\tau = t} = 0.$$

The Lagrange multiplier, therefore, can be defined in the following form: $\lambda_1(\tau) = \lambda_2(\tau) = -1$.
(3.3) Substituting from (3.3) into the correction functional equations (3.1) and (3.2) result the following iteration formulas:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t [u_{n\tau} - 0.5\,i\,u_{nxx} - i\,(|u_n|^2 + |v_n|^2)\,u_n]\,d\tau\,, \tag{3.4}$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t [v_{n\tau} - 0.5\,i\,v_{nxx} - i\,(|u_n|^2 + |v_n|^2)\,v_n]\,d\tau\,. \tag{3.5}$$

Using the following initial approximations:

$$u_0(x,t) = u(x,0) = a_0(1 - \varepsilon\cos(\sqrt{2}\alpha\,x))\,, \tag{3.6}$$

$$v_0(x,t) = v(x,0) = b_0(1+\varepsilon\cos(\sqrt{2}\alpha x)), \tag{3.7}$$

where $a_0$, $b_0$ are the initial amplitudes of the two perturbed periodic waves, respectively, $\varepsilon << 1$ is a small parameter which represents the strength of the perturbation and $\alpha$ is the wave number of the perturbation. We can obtain directly the components $u_1(x,t)$ and $v_1(x,t)$ of the solution:

$$u_1(x,t) = u_0(x,t) - (a_0 t[-i(a_0^2+b_0^2) + \varepsilon\cos(\sqrt{2}\alpha x)[3i a_0^2 - i b_0^2 - i\alpha^2 - \varepsilon i\cos(\sqrt{2}\alpha x)$$
$$[3a_0^2 - b_0^2 - (a_0^2+b_0^2)\varepsilon\cos(\sqrt{2}\alpha x)]]]);$$

$$v_1(x,t) = v_0(x,t) - (b_0 t[-i(a_0^2+b_0^2) + \varepsilon\cos(\sqrt{2}\alpha x)[i a_0^2 - 3i b_0^2 + i\alpha^2 + \varepsilon i\cos(\sqrt{2}\alpha x)$$
$$[a_0^2 - 3b_0^2 - (a_0^2+b_0^2)\varepsilon\cos(\sqrt{2}\alpha x)]]]);$$

The rest of components of the iterative formulas (3.4) and (3.5) were obtained in the same manner using the Mathematica Package. The numerical behavior of the solutions by VIM for different time values $0 \le t \le 20$ in the region $0 \le x \le 115$ are shown in Figure (1), where the approximated wave solutions $|u(x,t)|$ is represented in the top, and $|v(x,t)|$ is represented in the bottom of Fig.(1). The numerical results are obtained by using two terms only from the iterative formulas (3.4)-(3.5) where $a_0 = 0.08$, $b_0 = 0.1$, $\alpha = 0.05$, $\varepsilon = 0.05$. We achieved a very good approximation for the solution of the system. It is evident that the overall errors can be made smaller by adding new terms from the iteration formulas.

## 2.2. Implementation of ADM

To apply ADM for the focusing Manakov system in one dimension, first we rewrite the equations (1.1)-(1.2) in the following operator form:

$$L_t u = 0.5\, i\, u_{xx} + i(|u|^2 + |v|^2)u, \tag{3.8}$$

$$L_t v = 0.5\, i\, v_{xx} + i(|u|^2 + |v|^2)v, \tag{3.9}$$

where the notation $L_t = \frac{\partial}{\partial t}$, symbolize, linear differential operator. By using the inverse operator, we can re-write (3.8)-(3.9) in the following form:

$$u(x,t) = u(x,0) + 0.5\, i\; L_t^{-1}[u_{xx}] + i\, L_t^{-1}[N_1(u,v)], \tag{3.10}$$

$$v(x,t) = v(x,0) + 0.5\, i\; L_t^{-1}[v_{xx}] + i\, L_t^{-1}[N_2(u,v)], \tag{3.11}$$

where the inverse operator $L_t^{-1}$ is defined by $L_t^{-1}(\bullet) = \int_0^t (\bullet)\, dt$ and the nonlinear terms $N_1(u,v)$ and

$N_2(u,v)$ are defined by: $N_1(u,v) = (|u|^2 + |v|^2)\,u$, $\quad N_2(u,v) = (|u|^2 + |v|^2)\,v$, (3.12)

The solutions $u(x,t)$ and $v(x,t)$ can be decomposed by an infinite series as follows:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)\,, \qquad v(x,t) = \sum_{i=0}^{\infty} v_i(x,t)\,, \tag{3.13}$$

where $u_i(x,t)$ and $v_i(x,t)$ are the components of $u(x,t)$ and $v(x,t)$ that will elegantly determined, and the nonlinear terms decomposed by the following infinite series:

$$N_k(u,v) = \sum_{m=0}^{\infty} A_{km}\,, \qquad k = 1,2. \tag{3.14}$$

where $A_{km}$, $k = 1,2$ are called Adomian's polynomials and definite by:

$$A_{km} = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N_k\left(\sum_{i=0}^{m} \lambda^i u_i, \sum_{i=0}^{m} \lambda^i v_i\right)\right]_{\lambda=0}, \qquad i \geq 0 \,. \tag{3.15}$$

From the above considerations, the decomposition method defines the components $u_i$ and $v_i$ for $i \geq 0$, by the following recursive relationships:

$$u_0(x,t) = u(x,0)\,, \quad u_{n+1}(x,t) = 0.5\,i\,L_t^{-1}(u_{nxx}) + i\,L_t^{-1}(A_{1n})\,, \quad n \geq 0. \tag{3.16}$$

$$v_0(x,t) = v(x,0)\,, \quad v_{n+1}(x,t) = 0.5\,i\,L_t^{-1}(v_{nxx}) + i\,L_t^{-1}(A_{2n})\,, \quad n \geq 0. \tag{3.17}$$

This will enable us to determine the components $u_n$ and $v_n$ recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparisons purpose, we construct the solutions $u(x,t)$ and $v(x,t)$: $\lim_{n\to\infty} \Psi_n = u(x,t)$, $\lim_{n\to\infty} \Theta_n = v(x,t)$, where

$$\Psi_n(x,t) = \sum_{i=0}^{n-1} u_i(x,t), \;\; \Theta_n(x,t) = \sum_{i=0}^{n-1} v_i(x,t)\,, \; i \geq 0. \tag{3.18}$$

Now, we can obtain the first Adomian's polynomials of $A_{km}$, $k = 1,2$ using equation (3.15) as follows:

$$\begin{aligned}
&A_{10} = q\,u_0(|u_0|^2 + |v_0|^2)\,, \qquad A_{20} = q\,v_0(|u_0|^2 + |v_0|^2)\,,\\
&A_{11} = q\,u_1(|u_0|^2 + |v_0|^2) + 2q\,u_0(|u_0||u_1| + |v_0||v_1|)\,,\\
&A_{21} = q\,v_1(|u_0|^2 + |v_0|^2) + 2q\,v_0(|u_0||u_1| + |v_0||v_1|)\,,
\end{aligned} \tag{3.19}$$

and so on. Staring with the following initial approximation:

$$u_0(x,t) = a_0(1-\varepsilon\cos(\sqrt{2}\alpha x)), \qquad v_0(x,t) = b_0(1+\varepsilon\cos(\sqrt{2}\alpha x)),$$

and by recurrence formulas (3.16), (3.17) we can obtain directly the components $u_1, v_1$ of the solution:

$$u_1(x,t) = a_0 t[i\,(a_0^2+b_0^2)+\varepsilon\cos(\sqrt{2}\,\alpha x)[-3i\,a_0^2+i\,b_0^2+i\alpha^2+i\,\varepsilon\cos(\sqrt{2}\,\alpha x)[3a_0^2-b_0^2-(a_0^2+b_0^2)\varepsilon\cos(\sqrt{2}\,\alpha x)]]],$$

$$v_1(x,t) = b_0 t[i\,(a_0^2+b_0^2)+\varepsilon\cos(\sqrt{2}\,\alpha x)[-i\,a_0^2+3i\,b_0^2-i\alpha^2-i\,\varepsilon\cos(\sqrt{2}\,\alpha x)[a_0^2-3b_0^2-(a_0^2+b_0^2)\varepsilon\cos(\sqrt{2}\,\alpha x)]]],$$

The rest of components of the iterative formulas (3.16) and (3.17) were obtained in the same manner using the Mathematica Package. The numerical behavior of the solutions by ADM are shown in the same Figure (1), The numerical results are obtained by using two terms only from the iterative formulas (3.16)-(3.17). We achieved a very good approximation for the solution of the system. It is evident that the overall errors can be made smaller by adding new terms from the iteration formulas.

## 3. Conserved Quantities in One Space Variable

To illustrate whether the proposed methods lead to higher accuracy in one dimension, we will use the same procedure as in Sun and Qin [17] which emphasize that a good numerical scheme should have excellent long-time numerical behavior, as well as energy conservation property. To monitor the accuracy of the VIM and ADM, we consider the following two conserved quantities:

$$E(u) = \int_{-s/2}^{s/2} |u(x,t)|^2\,dx \quad \text{and} \quad E(v) = \int_{-s/2}^{s/2} |v(x,t)|^2\,dx \tag{3.20}$$

where $s = (2\pi/\alpha) = 40\pi$ ( for $\alpha = 0.05$ ) is the spatial period of the solution [20]. Table 1 shows the quantity $E(u)+E(v)$ for various times by VIM and ADM. The nearly constant values of $E(u)+E(v)$ show that both methods are working well.

Table 1.

| Time | VIM: $E(u)+E(v)$ | ADM: $E(u)+E(v)$ |
|---|---|---|
| 2 | 2.05599 | 2.05599 |
| 4 | 2.06258 | 2.06258 |
| 6 | 2.07356 | 2.07356 |
| 8 | 2.08894 | 2.08894 |
| 10 | 2.10871 | 2.10871 |
| 12 | 2.13287 | 2.13287 |
| 14 | 2.16143 | 2.16143 |
| 16 | 2.19438 | 2.19438 |

Now, by using the stability analysis suggested by Tan and Boyd [20], the wave solution is linearly stable

only if the perturbation wave number $\alpha$ is above the critical value $\alpha_c = \sqrt{2(a_0^2 + b_0^2)}$ ; otherwise the
wave solution is unstable. By choosing the constants $a_0 = 0.08$ , $b_0 = 0.1$, $\alpha = 0.05$ , we find that $\alpha_c$=0.181108, therefore, the wave solution in this case is unstable. The amplitude of u and v undergoes oscillations between the near-uniform state and the one-hume state ( see Fig. 1).

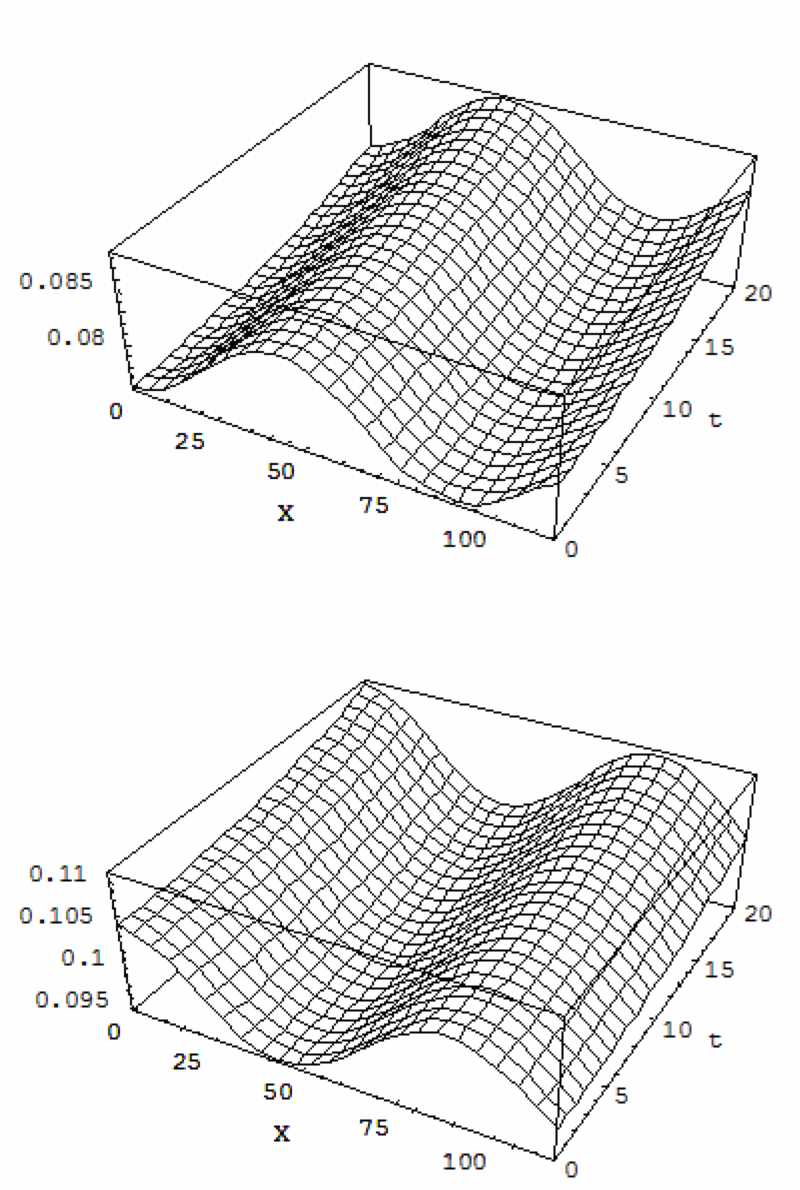


**Figure 1:** Long-time evolution of the wave solution |u(x, t)| and |v(x, t)|

## 4. Conclusions
The VIM and the ADM were used to find numerical solutions of the focusing Manakov systems of coupled nonlinear Schrödinger equations in one space variable. It may be concluded that there are

many advantages of these methods, the main advantages are the fast convergence to the solution, does not require discretizations of space and time variables, no need to solve nonlinear system of equations as in finite element method and finite difference method, then, no necessity of large computer memory. The accuracy of both methods is verified for the focusing Manakov systems by ensuring that the conserved quantities remain almost constant. A clear conclusion can be draw from the numerical results that the VIM is easier and faster than the ADM, moreover it overcomes the difficulty arising in calculating Adomian's polynomials in ADM.

**References**

[1] S. Abbasbandy and M. T. Darvishi 2005 Applied Mathematics and Computation **163** 1265
[2] M. A. Abdou and A. A. Soliman2005 *Physica D* **211** 1
[3] M. A. Abdou and A. A. Soliman2004 *J. Comput. Appl. Math.* 1
[4] G. Adomian 1989 *Nonlinear Stochastic Systems and Applications to Physics* (Dordrecht: Kluwer Academic Publishers)
[5] R. K. Bhattacharyya and R. K. Bera2004 *Applied Mathematics Letters* **17** 703
[6] Q. Chang. J. Erhui and W. Sun1999 *J. Comp. Physics* **148** 397
[7] S. M. El-Sayed and D. Kaya 2004 *Applied Mathematics and Computation* **158** 101
[8] J. H. He 2000 Applied Mathematics and Computation **114** 115
[9] J. H. He 1999 *Int. J. Non-Linear Mech.* **34** 699
[10] J. H. He and Xu-Hong Wu 2006 Chaos, Solitons and Fractals **29** 108
[11] D. Kaya and S. M. El-Sayed 2003 Physics Letters A **313** 82
[12] D. Kaya and I. E. Inan 2005 *Applied Mathematics and computation* **161** 1015
[13] S. Momani and Abuasad 2005 *Chaos, Solitons and Fractals* **27** 1119
[14] Otis C. 1995 *Physica D* **82** 1
[15] J.A.E. Rocha, Numerical short-time behavior of Manakov system, preprint, available at http://math.arizona.edu/~jaer/files/PDEFEfiles.html.
[16] A. Soufyane and M. Boulmalf 2005 *Applied Mathematics and Computation* **162** 687
[17] J. Q. Sun and M. Z. Qin 2003 *Comput. Phys. Commun.* **155** 221
[18] N. H. Sweilam, 2007 *J. Comput. Appl. Math.* **207** 64
[19] N. H. Sweilam and M. M. Khader 2007 *Chaos, Solitons and Fractals* **32** 145
[20] B. Tan and J. P. Boyd 2005 *Chaos, Solitons and Fractals* **12** 721
[21] S. C. Tsang and K. W. Chow 2004 *Mathematic and Computers in Simulation* **66** 551
[22] P. Vadas and S. Olek 2000 *Int. J. Heat and Mass Transfer* **43** 1715
[23] Wazwaz A. M. 1997 *Appl. Math.Comp*.**81** 265
[24] V. E. Zakharov and E.I. Schulman 1982 *Physica D* **4** 270